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von Neumann equations with time-dependent Hamiltonians and supersymmetric quantum mechanics

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Starting with a time-independent Hamiltonian h and an appropriately chosen solution of the von Neumann equation $i\dot{\rho}(t)=[h,\rho(t)]$ we construct its binary-Darboux partner $h_1(t)$ and an exact scattering solution of $i\dot{\rho}_1(t)=[h_1(t),\rho_1(t)]$, where $h_1(t)$ is time dependent and not isospectral to h. The method is analogous to supersymmetric quantum mechanics but is based on a different version of a Darboux transformation. We illustrate the technique by the example where h corresponds to a one-dimensional harmonic oscillator. The resulting $h_1(t)$ represents a scattering of a solitonlike pulse on a three-level system.

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One of the main ideas of supersymmetric (SUSY) quantum mechanics (QM) can be summarized as follows [1]. Assume we know a ground state $|\psi_0\rangle$ of a stationary Schrödinger equation (SE)

$$H|\psi_0\rangle=(H_{\rm kin}+V)|\psi_0\rangle=E_0|\psi_0\rangle, \tag{1}$$

with some V and E_0. Using $|\psi_0\rangle$ we construct an ''annihilation'' operator $A=A(\psi_0)$ satisfying $A|\psi_0\rangle=0$ and $H-E_0=A^\dagger A$. Now define $|\psi_1\rangle:=A|\psi\rangle$ (here $|\psi\rangle$ is any eigenstate of H linearly independent of $|\psi_0\rangle$, with eigenvalue E) and $H_1=AA^\dagger=H_{\rm kin}+V_1$. H_1 is the so-called SUSY partner Hamiltonian of H. Then, using $AH=AA^\dagger A=H_1A$, one finds that

$$(H_{\rm kin}+V_1)|\psi_1\rangle=E|\psi_1\rangle. \tag{2}$$

In a single step we have produced a new potential V_1 and one solution of the corresponding stationary SE.

The map $V\to V_1$ is known to be a particular example of a *Darboux transformation* (DT) [2]. All DT's transform a ''potential'' V into V_1 and simultaneously generate an ''annihilation'' operator $A(\psi_0)$ satisfying $A(\psi_0)\psi_0=0$, where ψ_0 is a solution of some partial differential linear equation associated with V. The physical interpretation of such an abstract ''potential'' depends on the problem.

SUSY QM deals with *linear* SE, and for this reason the density matrix generalization is not interesting: H_1 can be inserted either into the SE or into the von Neumann equation (vNE) $i\dot{\rho}=[H_1,\rho]$. However, the vNE has a structure which is algebraically different from that of the SE, and therefore allows for different DT's. A candidate is the so-called binary DT (BDT) originally constructed in Ref. [3], and applied to optical soliton equations. Quite recently the technique was applied to Yang-Mills equations [4] and nonlinear vNE [5,6]. A tutorial introduction to density-matrix applications was given in Ref. [7]. There are formal analogies between the BDT and the ''dressing method'' of Novikov *et al.* [8], but technically the two procedures are inequivalent (for a discussion, cf. Refs. [4,6]).

The purpose of the paper is to show that the BDT leads to a new kind of SUSY-type QM for density matrices which does not have a counterpart in SUSY QM based on SE. At an intermediate stage of the construction we solve the nonlinear vNE

$$i\dot{\rho}=[H,\rho^2], \tag{3}$$

where H is a time-independent Hamiltonian. The set of solutions of Eq. (3) contains all the pure states of standard QM since for $\rho^2=\rho$ (3) reduces to the linear vNE. For $\rho^2\neq\rho$ there exist at least two more classes of solutions. One of them occurs for ρ's satisfying either $\rho^2-a\rho=0$ with $a\in\mathbb{R},a\neq 1$, or a weaker condition $[H,\rho^2-a\rho]=0$ (now $a=1$ is acceptable). In both cases $\rho(t)=e^{-iaHt}\rho(0)e^{iaHt}$. The second class is of the form $\rho(t)=e^{-iaHt}\rho_{\rm int}(t)e^{iaHt}$ with $\rho_{\rm int}(-\infty)\neq\rho_{\rm int}(+\infty)$. These additional solutions, here called self-scattering (SS) solutions, are fundamental to our construction because of the following property: Each SS solution of the *nonlinear* vNE (3) with a time-*independent* H is simultaneously a scattering solution of a *linear* vNE with a time-*dependent* Hamiltonian $h_1(t)$. Both the SS solution and the new Hamiltonian are algebraically constructed in terms of the BDT. The construction does not make use of supercharges, and for this reason the resulting partner Hamiltonians will be termed the binary-Darboux (BD) partners.

The BDT method of solving Eq. (3) was described in Refs. [5,6]. We start with the family of Lax pairs, parametrized by $\omega\in\mathbb{C}$,

$$z_\omega|\psi_\omega\rangle=(\rho-\omega H)|\psi_\omega\rangle, \tag{4}$$

$$i|\dot{\psi}_\omega\rangle=(H\rho+\rho H-\omega H^2)|\psi_\omega\rangle, \tag{5}$$

where z_ω is a complex eigenvalue. Pair (4) and (5) is here the analog of Eq. (1); ρ and $h=H\rho+\rho H$ play the role of the ''potentials.''

The connection of Eqs. (4) and (5) with Eq. (3) is twofold. First, the necessary condition for the existence of $|\psi_\omega\rangle$ is given by Eq. (3). Now assume $|\psi_\mu\rangle=|\psi_\mu(t)\rangle$ is any solution of Eqs. (4) and (5) with $\omega=\mu$ and some z_μ. Denote by P_μ the projector on $|\psi_\mu\rangle$, and let $\lambda\in\mathbb{C}$ be another parameter, and ρ any solution of Eq. (3). Defining

$$\rho_1=\left(\mathbf{1}+\frac{\mu-\bar{\mu}}{\bar{\mu}}P_\mu\right)\rho\left(\mathbf{1}+\frac{\bar{\mu}-\mu}{\mu}P_\mu\right)=:U_\mu\rho U_\mu^\dagger \quad (6)$$

$$|\psi_{\lambda,1}\rangle=\left(\mathbf{1}-\frac{\mu-\bar{\mu}}{\lambda-\bar{\mu}}P_\mu\right)|\psi_\lambda\rangle=:A(\psi_\mu)|\psi_\lambda\rangle \quad (7)$$

we find (cf. Refs. [5,6])

$$z_\lambda|\psi_{\lambda,1}\rangle=(\rho_1-\lambda H)|\psi_{\lambda,1}\rangle, \quad (8)$$

$$i|\dot{\psi}_{\lambda,1}\rangle=(H\rho_1+\rho_1 H-\lambda H^2)|\psi_{\lambda,1}\rangle. \quad (9)$$

The ''Hamiltonians'' $\rho-\lambda H$ and $\rho_1-\lambda H$ possess the same eigenvalue z_λ, and their eigenvectors are related by the ''annihilation operator'' A [note that $A(\psi_\mu)|\psi_\mu\rangle=0$]. However, these are not the physical BD partners we are interested in. The BDT transforms the two ''potentials'' $\rho\to\rho_1$ and $h\to h_1$ in such a way that

$$i\dot{\rho}_1=[H\rho_1+\rho_1 H,\rho_1]=[h_1,\rho_1], \quad (10)$$

since this condition has to be satisfied whenever $|\psi_{\lambda,1}\rangle$ exists. The BD-transformed Lax pair (8) and (9) can be used to repeat the procedure: $\rho_1\to\rho_2$ and $h_1\to h_2$.

To explicitly show that the construction of h_1 is nontrivial, we have to make an assumption about the Hamiltonian H. We shall concentrate on the isospectral family of the one-dimensional (1D) harmonic oscillator (HO), since for Hamiltonians with equally spaced spectra a strategy leading to nontrivial solutions was worked out in detail in Ref. [5]. An alternative strategy was described in Ref. [6], and applied to a concrete example in Ref. [7]. In both cases the result is a SS solution.

We take the Hamiltonian $H=\epsilon N$, where ϵ is some parameter,

$$N=\sum_{n=0}^{\infty}(r+n)|r+n\rangle\langle r+n|, \quad (11)$$

and $r\in\mathbb{R}$ (e.g., for a 1D HO, $r=1/2$; for a 3D isotropic HO, $r=3/2$). In the Hilbert space spanned by $\{|r+n\rangle\}_{n=0}^{\infty}$, consider a 3D subspace spanned by three subsequent excited states $|k\rangle,|k+1\rangle$, and $|k+2\rangle$. It should be stressed that the same strategy can be applied to any H with discrete spectrum, provided there exist three eigenvalues of H satisfying $E_k-E_l=E_l-E_m$.

In order to obtain a SS solution $\rho_1(t)$, one has to start with an appropriate $\rho(t)$. The problem of how to select such a ρ was discussed in great detail in Ref. [5]. The fact that Eq. (15) does indeed solve Eq. (3) with H given by Eq. (11) can be verified by a straightforward calculation.

We consider a one-parameter family of solutions, parametrized by $\alpha\in\mathbb{R}$. Physically the parameter turns out to control the scattering process. Mathematically it parametrizes an initial condition for the solution of the Lax pair (4) and (5). We solve Eqs. (4) and (5) with

$$\rho(t)=e^{-i5Ht}\rho(0)e^{i5Ht}=:W_5\rho(0)W_5^\dagger, \quad (12)$$

$$\begin{aligned}\rho(0)=&\frac{5}{2}(|k\rangle\langle k|+|k+2\rangle\langle k+2|)\\&+\frac{5+\sqrt{5}}{2}|k+1\rangle\langle k+1|\\&-\frac{3}{2}(|k+2\rangle\langle k|+|k\rangle\langle k+2|),\end{aligned} \quad (13)$$

and $\mu=i/\epsilon$. For later purposes we have introduced the unitary operator $W_a(t)=e^{-iaHt}$. Equation (12) is a solution of Eq. (3), and therefore the necessary condition for the existence of $|\psi_{i/\epsilon}(t)\rangle$ is satisfied. $5H$ in Eq. (12) comes from $[H,\rho(0)^2]=5[H,\rho(0)]$, and the resulting equalities

$$i\dot{\rho}=[H\rho+\rho H,\rho]=5[H,\rho]=[h,\rho]. \quad (14)$$

$h=5H$ can be regarded as the first element of the pair of BD partner Hamiltonians we are going to find. The initial condition for Eqs. (4) and (5) is

$$\begin{aligned}|\psi_{i/\epsilon}(0)\rangle=&\frac{1}{\sqrt{1+\alpha^2}}|k+1\rangle\\&+\frac{\alpha}{\sqrt{1+\alpha^2}}\left(-i\sqrt{\frac{3+\sqrt{5}}{6}}|k\rangle\right.\\&\left.+\sqrt{\frac{2}{9+3\sqrt{5}}}|k+2\rangle\right).\end{aligned}$$

Inserting $P_{i/\epsilon}$, which projects on $|\psi_{i/\epsilon}(t)\rangle$, into Eq. (6) with $\mu=i/\epsilon$, and normalizing the resulting solution to obtain $\mathrm{Tr}\,\rho_1=1$, we finally obtain the density matrix

$$\rho_1(t)=\sum_{u,v=0}^{2}\rho_1(t)_{1+u,1+v}|k+u\rangle\langle k+v|, \quad (15)$$

where the matrix of coefficients in Eq. (15) is

$$\rho_1(t)_{\dots}=\frac{1}{15+\sqrt{5}}\begin{pmatrix}5 & \xi(t) & \zeta(t)\\ \bar{\xi}(t) & 5+\sqrt{5} & \xi(t)\\ \bar{\zeta}(t) & \bar{\xi}(t) & 5\end{pmatrix}, \quad (16)$$

with

$$\xi(t)=\frac{(2+3i-\sqrt{5}i)\sqrt{3+\sqrt{5}}\,\alpha}{\sqrt{3}(e^{\omega_0 t/5}+\alpha^2 e^{-\omega_0 t/5})}e^{i\omega_0 t}, \quad (17)$$

$$\zeta(t)=-\frac{9e^{2\omega_0 t/5}+(1+4\sqrt{5}i)\alpha^2}{3(e^{2\omega_0 t/5}+\alpha^2)}e^{2i\omega_0 t}, \quad (18)$$

and $\omega_0=10\epsilon/(15+\sqrt{5})$. Writing Eq. (15) as

$$\rho_1(t)=e^{-i\omega_0 Nt}\rho_{\mathrm{int}}(t)e^{i\omega_0 Nt}, \quad (19)$$

one finds, for $0<|\alpha|<\infty$,

$$\rho_{\rm int}(-\infty) = \frac{1}{15+\sqrt{5}}\begin{pmatrix} 5 & 0 & -\frac{1}{3}-\frac{4\sqrt{5}i}{3} \\ 0 & 5+\sqrt{5} & 0 \\ -\frac{1}{3}+\frac{4\sqrt{5}i}{3} & 0 & 5 \end{pmatrix};$$

$$\rho_{\rm int}(+\infty) = \frac{1}{15+\sqrt{5}}\begin{pmatrix} 5 & 0 & -3 \\ 0 & 5+\sqrt{5} & 0 \\ -3 & 0 & 5 \end{pmatrix}.$$

This shows that Eq. (15) is a SS solution.

The BD partner h_1 occurring in Eq. (10) is nonunique, and defined up to an operator commuting with ρ_1. This freedom is useful. Set

$$h_1 = (H+\epsilon c_1 \mathbf{1})\rho_1 + \rho_1(H+\epsilon c_1\mathbf{1}) + \epsilon c_2 \mathbf{1}, \tag{20}$$

with constants c_1 and c_2. Denoting $\sigma_{j,k} = |j\rangle\langle k|$, and using the above explicit solution, we find

$$h_1(t) = \tilde{H} + H_1(t), \tag{21}$$

where $\tilde{H} = \omega_0(N+c_1\mathbf{1}) + \epsilon c_2 \mathbf{1}$ and

$$\begin{aligned} H_1(t) = {} & \frac{\omega_0}{\sqrt{5}}(k+c_1+1)\sigma_{k+1,k+1} \\ & + \frac{\omega_0}{5}\left(k+c_1+\frac{1}{2}\right)[\xi(t)\sigma_{k,k+1} + \overline{\xi}(t)\sigma_{k+1,k}] \\ & + \frac{\omega_0}{5}(k+c_1+1)[\zeta(t)\sigma_{k,k+2} + \overline{\zeta}(t)\sigma_{k+2,k}] \\ & + \frac{\omega_0}{5}\left(k+c_1+\frac{3}{2}\right)[\xi(t)\sigma_{k+1,k+2} + \overline{\xi}(t)\sigma_{k+2,k+1}]. \end{aligned}$$

$\xi(t)$ and $\zeta(t)$ are essentially the Rabi frequencies. The nonuniqueness of h_1 was used again to extend the nonperturbed part of Eq. (21) beyond the 3D subspace. Our construction guarantees that Eq. (15) is a scattering solution of the corresponding time-dependent *linear* vNE $i\dot{\rho}_1 = [h_1(t), \rho_1]$. Let us note here that the dynamics of ρ_1 is related to $\rho(0)$ by the *unitary* transformation $U_{i/\epsilon}W_5$. In general, taking arbitrary U_μ and W_a, we can alternatively define the scattering Hamiltonian as

$$h_1 = i\dot{U}_\mu U_\mu^\dagger + aU_\mu H U_\mu^\dagger. \tag{22}$$

h_1 is a nontrivial scattering Hamiltonian provided $\rho_1(t)$ is a SS solution of Eq. (3).

Equation (21) represents a complicated time-dependent three-level perturbation of a HO. In order to better understand the kind of interaction we have produced, set $\alpha = 1$, $c_1+k+1=0$, $\epsilon c_2 = -\omega_0 c_1$, and $d = \sqrt{3+\sqrt{5}}(2+3i - \sqrt{5}i)/(2\sqrt{3})$. The Hamiltonian now reads

$$\begin{aligned} h_1(t) = {} & \omega_0 N - \frac{\omega_0 d e^{i\omega_0 t}}{10\cosh(\omega_0 t/5)}(\sigma_{k,k+1} - \sigma_{k+1,k+2}) \\ & - \frac{\omega_0 d^* e^{-i\omega_0 t}}{10\cosh(\omega_0 t/5)}(\sigma_{k+1,k} - \sigma_{k+2,k+1}). \end{aligned} \tag{23}$$

One can think of h_1 as describing a 1D HO located at $x=0$ and interacting with the well-known McCall-Hahn ''sech'' optical soliton [10]. Let us recall, however, that the result is more general and valid for any H with discrete spectrum provided the 3D subspace corresponds to three equally spaced eigenvalues. Taking different parameters in h_1 we obtain additional terms reminiscent of the ''sech-tanh'' pulse occurring in inhomogeneously broadened three-level media [11]. It is interesting that for $c_1+k+1 \neq 0$ the perturbation $H_1(t)$ contains a time-independent term proportional to $|k+1\rangle\langle k+1|$. Redefining the nonperturbed part by

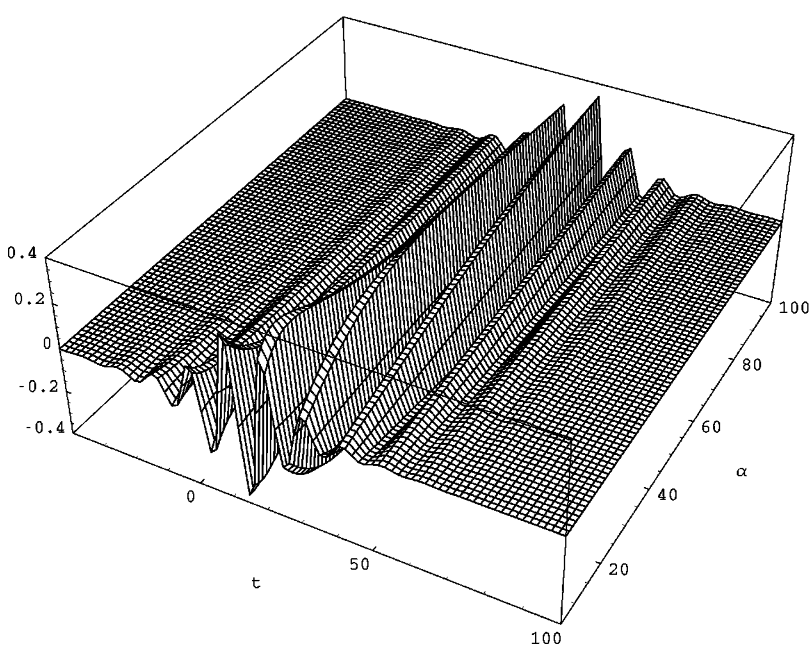

FIG. 1. $\langle x\rangle$ as a function of time and the parameter α, $5 \leqslant \alpha \leqslant 100$, which controls the initial condition. The moment of SS moves toward the future (past) as $|\alpha|$ grows (decreases). For $|\alpha| = \infty$ ($\alpha=0$) SS is shifted to $+(-)\infty$ (no scattering).

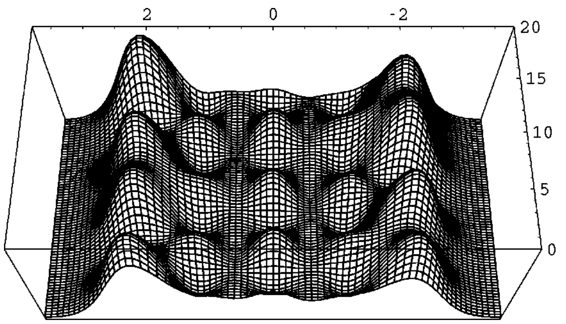

FIG. 2. Probability density in position space as a function of time for $k=2.5$, $\alpha=5$, $0<t<20$. The asymmetry of the probability density is responsible for the oscillation of $\langle x\rangle$ seen in Fig. 1.

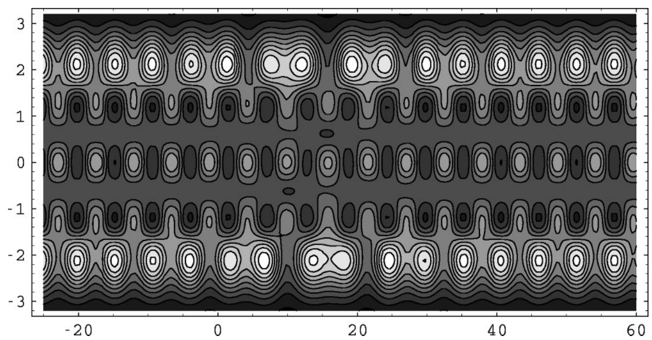

FIG. 3. Contour plot of the probability distribution from Fig. 2 for $-25<t<60$. The continuous transition between the two asymptotic states (with symmetric probability distributions) is clearly visible.

$$\tilde{H}'=\omega_0 N+\frac{\omega_0}{\sqrt{5}}(k+c_1+1)|k+1\rangle\langle k+1|, \qquad (24)$$

we break the equal spacing of the nonperturbed Hamiltonian, simultaneously detuning the highest and lowest levels from ω_0 and generating a transition with a doubled frequency $2\omega_0$.

A general property of Eq. (3) is the fact that $\langle H\rangle_n = \mathrm{Tr}\, H\rho^n$ are integrals of motion for any natural n and any solution ρ [9]. In particular, this implies that the sum of the perturbed eigenvalues of h_1 is time independent. The same holds for the average energy $\langle E\rangle = \mathrm{Tr}\, h_1(t)\rho_1(t)$. However, the eigenvalues themselves may be time dependent. For $c_1+k+1=0, c_2=0$, the eigenvalues of the restriction of h_1 to the 3D subspace are 0 and

$$\pm\frac{\omega_0}{5}\sqrt{25+4\frac{e^{2\omega_0 t/5}\alpha^2}{(e^{2\omega_0 t/5}+\alpha^2)^2}}.$$

This implies that the BD partners $h=5H$ and h_1 are not isospectral, a situation that may occur in higher-dimensional SUSY QM.

The figures illustrate properties of the scattering solutions. Figure 1 shows the average position of the 1D HO $\langle x\rangle = (1/\sqrt{2})\,\mathrm{Tr}\,\rho_1(a+a^\dagger)$ as a function of time and α. In the asymptotic regions the average is 0. For times where $\langle x\rangle \approx 0$ the dynamics is effectively given by

$$\rho_{\mathrm{in}}(t)=e^{-i\omega_0 N t}\rho_{\mathrm{int}}(-\infty)e^{i\omega_0 N t},$$

$$\rho_{\mathrm{out}}(t)=e^{-i\omega_0 N t}\rho_{\mathrm{int}}(+\infty)e^{i\omega_0 N t}.$$

As $|\alpha|$ grows the moment of SS is shifted toward the future. For $\alpha=0$ or $|\alpha|=\infty$ there is no scattering since ρ_{int} becomes time independent.

The asymptotic probability densities in position space $\varrho(x,t)=\langle x|\rho_1(t)|x\rangle$ are symmetric (implying $\langle x\rangle=0$) (Fig. 2) (also see Fig. 3) Such time-dependent probability distributions represent a new type of nonlinear effect.

The above effects can be extended to higher-dimensional subspaces. One of the possibilities is related to the ''weak superposition'' principle: For any family of solutions $\{\rho_k\}$ of Eq. (3) satisfying $\rho_k\rho_l=0$ for $k\neq l$, the combination $\rho(t) = \Sigma_k p_k\rho_k(p_k t)$ is also a solution of Eq. (3). One can generalize the procedure to many noninteracting HO's and consideration of systems with degeneracy, such as HO's with spin, leads to a nontrivial second iteration of the BDT: $\rho\to\rho_1\to\rho_2$ and $h\to h_1\to h_2$. Another possibility is related to the Yang-Mills (YM) case. The result of Ref. [4] shows that a class of YM equations can be integrated by BDT. The anti-self-dual YM case is algebraically related to Euler-Arnold equations [12], which are a particular case of Eq. (3) as discussed in Ref. [5].

Exactly solvable equations with time-dependent Hamiltonians are a rarity in quantum mechanics. The technique we have described leads to a broad class of such equations. The example we have discussed, in spite of its simplicity, shows the richness and efficiency of the method. The resulting three-level dynamics is highly nontrivial and physically interesting. We expect the method to prove useful in many branches of quantum physics.

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[1] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995), and references therein.

[2] V. B. Matveev and M. A. Salle, *Darboux Transfomations and Solitons* (Springer, Berlin, 1991).

[3] S. B. Leble and N. V. Ustinov, in *Nonlinear Theory and its Applications (NOLTA '93)* (NTA Research Society of IEICE, Tokyo, 1993), p. 547; A.A. Zaitsev and S.B. Leble, Rep. Math. Phys. **39**, 177 (1997); S.B. Leble, Comput. Math. Appl. **35**, 73 (1998).

[4] N.V. Ustinov, J. Math. Phys. **39**, 976 (1998).

[5] S.B. Leble and M. Czachor, Phys. Rev. E **58**, 7091 (1998).

[6] M. Kuna, M. Czachor, and S.B. Leble, Phys. Lett. A **255**, 42 (1999).

[7] M. Czachor, M. Kuna, S. B. Leble, and J. Naudts, in *New*

Insights in Quantum Mechanics, edited by H.-D. Doebner, S. T. Ali, M. Keyl, and R. F. Werner (World Scientific, Singapore, 1999); e-print quant-ph/9904110.

[8] S. P. Novikov, S. V. Manakov, L. P. Pitaevski, and V. E. Zakharov, *Theory of Solitons, the Inverse Scattering Method* (Consultants Bureau, New York, 1984).

[9] M. Czachor and J. Naudts, Phys. Rev. E **59**, 2497R (1999).

[10] S.L. McCall and E.L. Hahn, Phys. Rev. Lett. **18**, 908 (1967).

[11] A. Rahman and J.H. Eberly, Phys. Rev. A **58**, R805 (1998).

[12] L. J. Mason and N. M. J. Woodhouse, *Integrability, Self-Duality, and Twistor Theory* (Oxford University Press, Oxford, 1996).